

Effective Computation of the Sullivan model of a topological space and its rational homotopy type

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Outline

- 1) Introduction
- 2) (Minimal) Sullivan Algebras
- 3) The Sullivan model of a space
- 4) Properties of the Sullivan model
- 5) Applications and future work

1) Introduction

the rational homotopy type of a space is a simplified version of the homotopy type, where all homotopy groups are tensored by $\mathbb{Q}$

$$\boxed{\pi_n(x) \otimes \mathbb{Q}}$$

1) Introduction

	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
S^4	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_5$	$\mathbb{Z}_2$
S^5	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
S^6	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 1.. A subset of the homotopy groups of spheres S^4, S^5 and S^6

Something we know:

- Homotopy groups of S^k vanish until degree $k-1$ (included)
- $\pi_k(S^k) = \mathbb{Z}$
- π_n is abelian for $n \geq 2$
- $\pi_{2n}(S^{2n}) \otimes \mathbb{Q} = \pi_{4n-1}(S^{2n}) \otimes \mathbb{Q} = \mathbb{Q}$

1) Introduction

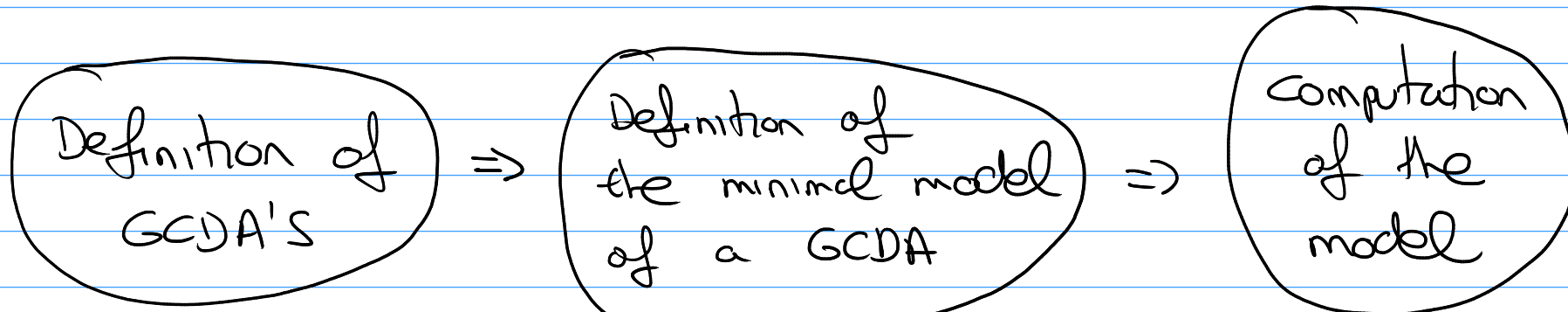
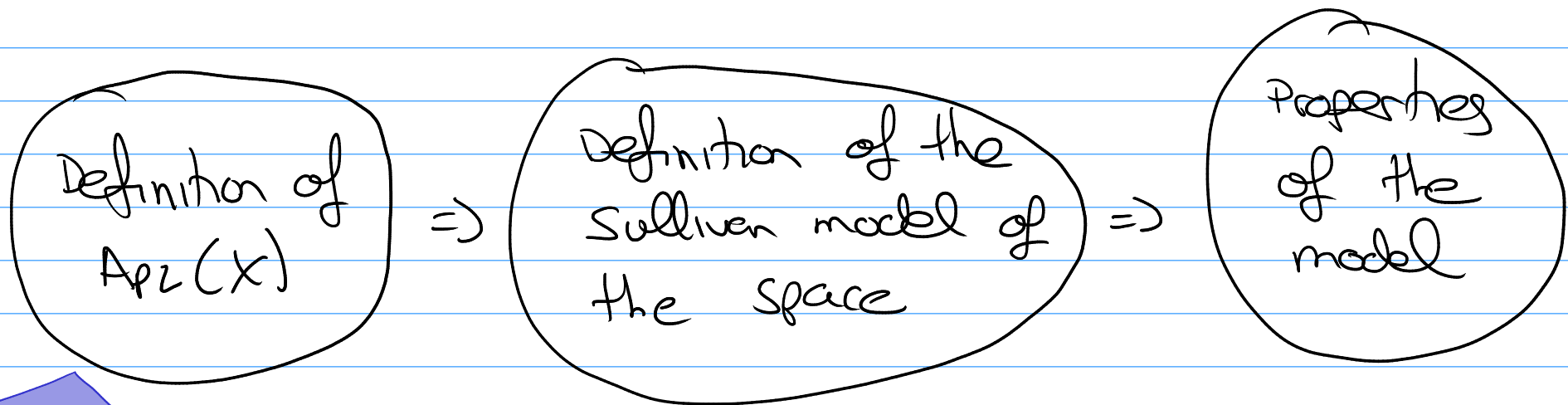
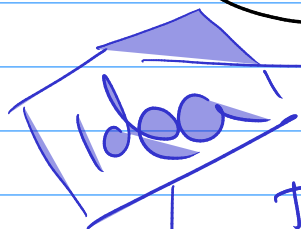
Despite the lost information, rational homotopy has the advantage of being constructive.

Due to Sullivan,

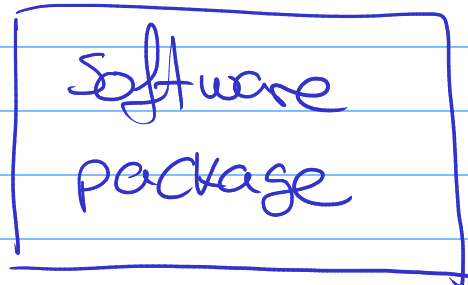
$$\boxed{X \rightsquigarrow M_x}$$

$$\left\{ \begin{array}{l} \text{Rational homotopy} \\ \text{types of spaces} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of minimal Sullivan} \\ \text{models} \end{array} \right\}$$

Structure of the talk



topological space $\Rightarrow$



$\Rightarrow$ Sullivan model

Definition of a GCDA

A graded commutative differential algebra (or GCDA), A , over a ring R , is a graded R -algebra, $A = \bigoplus_{i=0}^{\infty} A^i$, together with an R -linear map $d_A: A \rightarrow A$ that satisfies:

- $xy \in A^{i+j}, \forall x \in A^i, \forall y \in A^j$

- $xy = (-1)^{ij}yx, \forall x \in A^i, \forall y \in A^j$

- $d_A^2 = 0$

- $d_A(x) \in A^{i+1}, \forall x \in A^i$

- $d_A(xy) = d_A(x) \cdot y + (-1)^i x d_A(y), \forall x \in A^i, \forall y \in A$

From now on,
every algebra is
a GCDA

(Relative) Sullivan algebras

Def: A relative Sullivan algebra is a commutative cochain algebra

$$(B \otimes \Lambda V, d)$$

with the following properties:

- the "base algebra" $(B, d) = (B \otimes 1, d)$ is a subcochain algebra and connected, i.e., $H^0(B, d) = \mathbb{Q}$
- the algebra ΛV is connected and built upon a graded vector space $1 \otimes V = V = \{V^p\}_{p \geq 1}$

(Relative) Sullivan algebras

- there exists a filtration

$$V = \bigcup_{k=0}^{\infty} V(k)$$

of graded subspaces

$$V(0) \subseteq V(1) \subseteq V(2) \subseteq \dots \subseteq V$$

such that

$$d: V(0) \longrightarrow B$$

and

$$d: V(k) \longrightarrow B \otimes \Lambda(V(k-1)), \text{ for } k \geq 1$$

(Relative) Sullivan algebras

A Sullivan algebra is a relative Sullivan algebra
with $(B, d) = (\mathbb{Q}, 0)$

A relative Sullivan algebra is minimal if

$$\text{Im } d \subseteq B^{\geq 1} \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V$$

For a Sullivan algebra, this is equivalent to

$$\text{Im } d \subseteq \Lambda^{\geq 2} V$$

(Relative) Sullivan algebras: Examples

1) A CDGA that is not a Sullivan algebra: [Monero & Marco] 2020

Let A be a CDGA generated by $\langle e_1, \dots, e_7 \rangle$

where $|e_i| = 1$, for $i \leq 6$

$$|e_7| = 2$$

and the differential is given by:

$$\begin{aligned} d(e_1) &= -e_1 \wedge e_6 \\ d(e_2) &= -e_2 \wedge e_6 \\ d(e_3) &= -e_3 \wedge e_6 \end{aligned}$$

$$d(e_4) = -e_5 \wedge e_6$$

$$d(e_5) = d(e_6) = d(e_7) = 0$$

It is not a Sullivan algebra because the differentials of e_1, e_2 and e_3 are given as a product of themselves.

Sullivan algebras: Examples

2) A Sullivan algebra (not relative
not minimal)

$$(\Lambda(y, t), d) \quad \begin{array}{ll} |y| = 3 & dy = 0 \\ |t| = 2 & dt = y \end{array}$$

It is a Sullivan algebra
but it is not minimal
because it is not satisfied
the condition $\text{Im } d \subseteq \Lambda^{\geq 2} V$

3) A minimal Sullivan algebra

$$(\Lambda_2(x, y, z), d) \quad \begin{array}{l} |x| = |y| = |z| = 1 \\ dx = dy = 0 \\ dz = xy \end{array}$$

Definition of the minimal model of a GCDA

Let (A, d_A) be a connected ($H^0(A) = \mathbb{Q}$) GCDA.

Def: A minimal model of (A, d_A) is a minimal Sullivan algebra (M, d_M) , together with a quasi-isomorphism of
(Induces $\cong$ in cohomology)

$$\varphi: M \longrightarrow A$$

$$\varphi^*: H^*(M) \xrightarrow{\cong} H^*(A)$$

thm. Every connected GCDA, A , has a minimal model (M_A, φ_A)

thm. the minimal model of a GCDA is unique up to isomorphism

The algorithm for the minimal model

Let (A, d_A) be a connected ($H^0(A, d_A) = \mathbb{Q}$) commutative cochain algebra

$$\varphi: (M, d_M) \longrightarrow (A, d_A)$$

Step 1) Find the first $i \geq 1$ for which $H^i(A, d_A) \neq 0$

Add generators x_j to M^i ($|x_j| = i$) in such a way that $\varphi_*^i: H^i(M, d_M) \longrightarrow H^i(A, d_A)$ is an isomorphism.

The algorithm for the minimal model

step 2)

if $\varphi_{\star}^{i+1}: H^{i+1}(M, d_M) \rightarrow H^{i+1}(A, d_A)$ is injective

$\Rightarrow$ go back to step 1 and repeat the process for degree $i+1$

else ($\varphi_{\star}^{i+1}$ not injective)

Add to M^i generators y_i in such a way
you make $\varphi_{\star}^{i+1}$ injective.

Example [Manera & Marco 2020]

Let (A, d_A) be the CDGA generated by $\langle e_1, \dots, e_7 \rangle$
with $|e_i| = 1$, $i \leq 6$, has no relations
 $|e_7| = 2$

and the differential is given by

$$d_A(e_1) = -e_1 \wedge e_6$$

$$d_A(e_2) = -e_2 \wedge e_6$$

$$d_A(e_3) = -e_3 \wedge e_6$$

$$d_A(e_4) = -e_5 \wedge e_6$$

$$d_A(e_5) = d_A(e_6) = d_A(e_7) = 0$$

the first cohomology groups are

$$H^1(A) = \mathbb{Q} \langle [e_5], [e_6] \rangle$$

$$H^2(A) = \mathbb{Q} \langle [e_4 \wedge e_5], [e_4 \wedge e_6], [e_7] \rangle$$

$$H^3(A) = \mathbb{Q} \langle [e_4 \wedge e_5 \wedge e_6], [e_5 \wedge e_7], [e_6 \wedge e_7] \rangle$$

Example

[Manero & Marco 2020]

So we start the algorithm for degree 1

$$\begin{array}{ccccc} e_6 & \xleftarrow{\varphi} & x'_0 & \xrightarrow{d_M} & 0 \\ e_5 & \xleftarrow{\quad} & x'_1 & \xrightarrow{\quad} & 0 \end{array}$$

At this point, $H^1(\varphi)$ is an isomorphism and $H^2(M)$ would be generated by the class $[x'_0 \wedge x'_1]$ but,

$$\varphi(x'_0 \wedge x'_1) = -e_5 \wedge e_6 \quad \text{which is trivial in cohomology} \\ (d(e_4) = -e_5 \wedge e_6)$$

So, we have to add a new generator in order to make this cohomology class trivial:

$$e_4 \xleftarrow{\varphi} y'_0 \xrightarrow{d_M} x'_0 \wedge x'_1$$

Example [Manero & Marco 2020]

Now we have $M = \langle x'_0, x'_1, y'_0 \rangle$ and we have to check again the new elements that have been created in higher degrees.

$$d_M(x'_0 \wedge y'_0) = d_M(x'_1 \wedge y'_0) = 0$$

$$\text{so } H_M^2 = \mathbb{Q} \langle [x'_0 \wedge y'_0], [x'_1 \wedge y'_0] \rangle$$

In that case, we have that

$$\left. \begin{array}{l} x'_0 \wedge y'_0 \longrightarrow e_6 \wedge e_4 \\ x'_1 \wedge y'_0 \longrightarrow e_5 \wedge e_4 \end{array} \right\} \begin{array}{l} \text{which are two representatives of} \\ \text{two elements of the basis of } H^2(A) \end{array}$$

so the map φ_* is injective for degree 2

Example

[Manero & Marco 2020]

$$H^1(A) = \mathbb{Q} \langle [e_5], [e_6] \rangle$$

$$H^2(A) = \mathbb{Q} \langle [e_4 \wedge e_5], [e_4 \wedge e_6], [e_7] \rangle$$

$$H^3(A) = \mathbb{Q} \langle [e_4 \wedge e_5 \wedge e_6], [e_5 \wedge e_7], [e_6 \wedge e_7] \rangle$$

$$H^1(M) = \langle [x'_0], [x'_1] \rangle$$

$$H^2(M) = \langle [x'_0 \wedge y'_0], [x'_1 \wedge y'_0], [x_0^2] \rangle$$

$$H^3(M) = \langle [x'_0 \wedge x'_1 \wedge y'_0], [x'_0 \wedge x_0^2], [x'_1 \wedge x_0^2] \rangle$$

Now we continue creating the isomorphism in cohomology at degree 2. For that, we add to M a generator

$$e_7 \xleftarrow{\varphi} x_0^2 \xrightarrow{d_M} 0$$

and we have an isomorphism at $H^2(\varphi)$

But also we have an isomorphism at degree 3 because

$$d(x'_0 \wedge x'_1 \wedge y'_0) = 0$$

$$d(x'_0 \wedge x_0^2) = 0$$

$$d(x'_1 \wedge x_0^2) = 0$$

$$\varphi(x'_0 \wedge x'_1 \wedge y'_0) \longrightarrow e_6 \wedge e_5 \wedge e_4$$

$$\varphi(x'_0 \wedge x_0^2) \longrightarrow e_6 \wedge e_7$$

$$\varphi(x'_1 \wedge x_0^2) \longrightarrow e_5 \wedge e_7$$

Example: the minimal model of $(H^*(S^n; \mathbb{Q}), d=0)$ for $n \geq 2$

$$H^q(S^n, \mathbb{Q}) = \begin{cases} \langle h^q \rangle & \text{if } q = \{0, n\} \\ 0 & \text{otherwise} \end{cases} \quad \text{with } (h^n)^2 = 0$$

First, consider 'n' odd:

$$h^n \xleftarrow{\varphi} x_0^n \xrightarrow{d_n} 0$$

as $|x_0^n|$ is odd, $(x_0^n)^2 = 0$ so we have

that $\boxed{M \cong H^*(S^n)}$

Example: the minimal model of $H^*(S^n)$

Consider now the case 'n' even

$$h^n \xleftarrow{\varphi} x_0^n \xrightarrow{d_M} 0$$

Now we have an isomorphism at degree 'n'

but because 'n' is even, we have

$$H^{2n}(M) = \mathbb{Q} \langle [x_0^n]^2 \rangle$$

$$\text{but } H^{2n}(S^n) = 0$$

So we need to add a generator of degree $2n-1$
in order to make this cohomology trivial

$$0 \xleftarrow{\varphi} y_0^{2n-1} \xrightarrow{d_M} (x_0^n)^2$$

$$\text{So } \boxed{M = \mathbb{Q} \langle x_0^n, y_0^{2n-1} \rangle}.$$

The Sullivan model of a space

We assume that our topological space can be expressed as a simplicial complex / set.

the idea: we are going to construct an algebra

for that reason, we need

1) A topological space expressed as a simplicial set

2) the algebra of polynomial differential forms

3) A way to relate both structures

The Sullivan model of a space

Def: A simplicial object K with values in a category $\mathcal{C}$ is a sequence $\{K_n\}_{n \geq 0}$ of objects in $\mathcal{C}$, together with $\mathcal{C}$ -morphisms:

$$\boxed{d_i: K_{n+1} \rightarrow K_n, \quad 0 \leq i \leq n+1}$$

$$\boxed{s_j: K_n \rightarrow K_{n+1}, \quad 0 \leq j \leq n}$$

that satisfy the identities:

$$d_i d_j = d_{j-1} d_i, \quad i < j$$

$$s_i s_j = s_{j+1} s_i, \quad i \leq j$$

$$d_i s_j = \begin{cases} s_{j+1} d_i, & i < j \\ \text{id}, & i = j+1 \\ s_j d_{i-1}, & i > j+1 \end{cases}$$

Examples

- Simplicial sets
- Simplicial cochain algebras

The Sullivan model of a space

Def: A simplicial morphism $\phi: L \rightarrow K$ between two such simplicial objects is a sequence of d -morphisms ϕ_n , commuting with the faces and degeneracies.

$$\begin{array}{ccccc} L_{n-1} & \xrightleftharpoons[s_d]{\partial_i} & L_n & \xrightleftharpoons[s_j]{\partial_i} & L_{n+1} \\ \downarrow \phi_{n-1} & & \downarrow \phi_n & & \downarrow \phi_{n+1} \\ K_{n-1} & \xrightleftharpoons[s_j]{\partial_i} & K_n & \xrightleftharpoons[s_j]{\partial_i} & K_{n+1} \end{array}$$

the structure $A(K)$

Let K be a simplicial set, and let $A = \{A_n\}_{n \geq 0}$ be a simplicial cochain complex or a simplicial cochain algebra.

then,

$$A(K) = \{A^p(K)\}_{p \geq 0}$$

is the cochain complex (or cochain algebra) defined as follows:

$$K = \{K_n\}_{n \geq 0}$$

$$\hookrightarrow K_{n-1} \hookrightarrow K_n \hookrightarrow K_{n+1} \hookrightarrow$$

$$A = \{A_n\}_{n \geq 0}$$

$$\begin{array}{ccccc} A_{n-1}^{p-1} & \hookrightarrow & A_n^{p-1} & \hookrightarrow & A_{n+1}^{p-1} \\ \downarrow d & & \downarrow d & & \downarrow d \\ A_{n-1}^p & \hookrightarrow & A_n^p & \hookrightarrow & A_{n+1}^p \\ \downarrow d & & \downarrow d & & \downarrow d \\ A_{n-1}^{p+1} & \hookrightarrow & A_n^{p+1} & \hookrightarrow & A_{n+1}^{p+1} \end{array}$$

the structure $A(k)$

- $A^p(k)$ is the set of simplicial set morphisms from k to A^p .

thus, an element $\Phi \in A^p(k)$ is a mapping that assigns to each n -simplex $\sigma \in k_n$, an element $\Phi_\sigma \in (A^p)_n$ such that

$$\Phi(\partial_i \sigma) = \partial_i (\Phi(\sigma))$$

$$\Phi(s_j \sigma) = s_j (\Phi(\sigma))$$

the structure $A(k)$

- Addition, scalar multiplication and the differential are given by:

$$(\Phi + \Psi)(\sigma) = \Phi(\sigma) + \Psi(\sigma)$$

$$(\lambda \Phi)(\sigma) = \lambda \Phi(\sigma)$$

$$(d\Phi)(\sigma) = d(\Phi(\sigma))$$

- If A is a simplicial cochain algebra, multiplication in $A(k)$ is given by

$$(\Phi \cdot \Psi)(\sigma) = \Phi(\sigma) \cdot \Psi(\sigma)$$

the simplified commutative cochain algebra A_p

Consider the free graded commutative algebra

$$\Lambda(t_0, \dots, t_n, y_0, \dots, y_n)$$

$$\deg(t_i) = 0$$

$$\deg(y_i) = 1$$

$$\text{the differential} = \begin{cases} d(t_i) = y_i \\ d(y_i) = 0 \end{cases}$$

the simplicial commutative cochain algebra A_{PL}

the simplicial cochain algebra

$$A_{PL} = \{ (A_{PL})_n \}_{n \geq 0}$$

is defined as:

- the cochain algebra $(A_{PL})_n$ is given by:

$$(A_{PL})_n = \frac{N(t_0, \dots, t_n, y_0, \dots, y_n)}{\langle \sum t_i - 1, \sum y_i \rangle}$$

- Faces and degeneracies:

$$\partial_i = t_k \rightarrow \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases}$$

$$s_j = t_k \rightarrow \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j \end{cases}$$

the Sullivan model of a space

Given a topological space X as a simplicial set, we define the Sullivan model of X , as the minimal model of $A_{PL}(X)$

When X is simply-connected, the minimal model give us information about the rational homotopy type of the space

$$\#M^p = \dim (CT_p(X) \otimes \mathbb{Q})$$

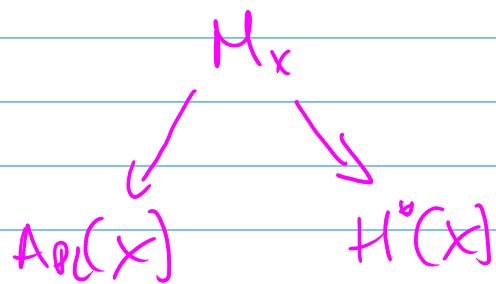
Applications

1) Isaia, Artal, González Pérez and Cármona

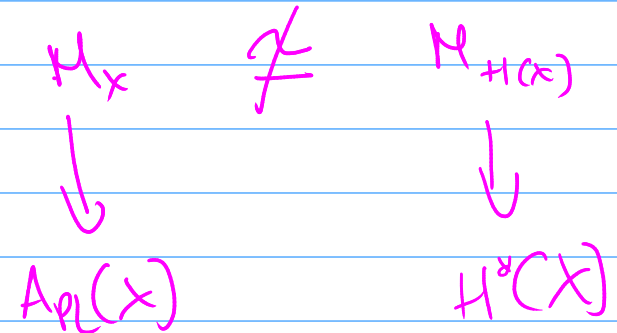
CW-Decomposition of Plane Algebraic Curves and
Milnor Fibers of Non-Isolated Quasi-Ordinary Singularities

2) Study of formality

Formal space



Non-formal space



Thank you

for your attention

References

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- 3) V. Manero, M. Marco, Effective computation of degree bounded minimal models of GCDAs, Journal of Software for Algebra and Geometry, 2020.